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Quantum Liouville Theory from a Diffeomorphism Chern-Simons Action

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Abstract

A Chern-Simons action written with Christoffel Symbols has a natural gauge symmetry of diffeomorphisms. This Chern-Simons action will induce a Wess-Zumio-Witten model on the boundary of the manifold. If we restrict the diffeomorphisms to chiral diffeomorphism, the Wess-Zumio-Witten model is equivalent to a quantum Liouville action.

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1 Introduction

Over the years, much work has gone into studying Liouville theory. In addition to being a natural candidate for two dimensional gravity, Liouville theory also arises naturally in non-critical dimensions in string theory. Some attempts have been made to further understand this theory by studying Chern-Simons theories on manifolds with boundary. Some of this work has been done in studies of topologically massive gravity [1] [2]. Topologically massive gravity contains a Chern-Simons term that is written in terms of spin connections. These spin connections possess an SO(2,1) gauge symmetry. This Chern-Simons action will induce an SO(2,1) Wess-Zumino-Witten (WZW) model on the boundary of the manifold. SO(2,1) is then homomorphic to SL(2,IR), and SL(2,IR) WZW is known to contain Liouville theory [3] [4] [5]. Thus, from a three dimensional gravity theory, we get a two dimensional gravity theory induced on the boundary of the manifold.

In this paper, I would like to extend this method in the following way. It has been shown that the Chern-Simons action expressed in terms of spin connections is equivalent to a Chern Simons action expressed in terms ofChristoffel symbols [6] (For simplicity I shall call this diffeomorphism Chern-Simons theory). Using this information as a starting point, I would like to complete the following diagram.

$$\begin{array}{ccccc} SO(2,1)\ CS & \longrightarrow & SL(2,\mathbb{R})\ WZW & \longrightarrow & Liouville \\ \downarrow & & & & \uparrow ? \\ Diffeo\ CS & & \xrightarrow{?} & & Diffeo\ WZW \end{array}$$

2 WZW from Chern-Simons Theory

I will start with the following Chern-Simons action on a 3-dimension manifold M with metric h.

$$\mathbf{S}_{CS} = \frac{k}{4\pi}\int_{\mathrm{M}} d^3x\ \varepsilon^{\mu\nu\lambda}\left(\Gamma_{\mu\ b}^{\ a}\partial_\nu\Gamma_{\lambda\ a}^{\ b} - \frac{2}{3}\Gamma_{\mu\ b}^{\ a}\Gamma_{\nu\ c}^{\ b}\Gamma_{\lambda\ a}^{\ c}\right) \tag{2.1}$$

Following Percacci's paper [6], $\Gamma_{\mu\ b}^{\ a}$ is defined in the following way. Let the metric tensor be written as

$$h_{\mu\nu} = \theta^a_{\ \mu}\theta^b_{\ \nu}\kappa_{ab}. \tag{2.2}$$

Then from (2.2), we see that

$$\begin{aligned} \Gamma_{\lambda\ b}^{\ a} &= \theta^e_{\ \lambda}\kappa^{af}\Gamma_{efb} \\ \Gamma_{efb} &= \frac{1}{2}\left(\theta_b^{\ \mu}\partial_\mu\kappa_{ef} + \theta_e^{\ \mu}\partial_\mu\kappa_{fb} - \theta_f^{\ \mu}\partial_\mu\kappa_{be}\right) + \frac{1}{2}\left(c_{efb} + c_{fbe} - c_{bef}\right) \\ c_{efb} &= \theta_{e\lambda}\left(\theta_f^{\ \mu}\partial_\mu\theta_b^{\ \lambda} - \theta_b^{\ \mu}\partial_\mu\theta_f^{\ \lambda}\right). \end{aligned} \tag{2.3}$$

For a fixed metric, Percacci shows that different values of κ and θ result in equivalent theories, up to a gauge transformation. So, if we choose $\kappa_{ab} = \delta_{ab}$ then from (2.2), we see that $\theta^a_{\ \mu}$

becomes the triad and $\Gamma_{\mu}{}^{a}{}_{b}$ becomes the spin connection. If we choose $\theta^{a}{}_{\mu} = \delta^{a}{}_{\mu}$ then from (2.2) and (2.3), we see that $\Gamma_{\mu}{}^{a}{}_{b}$ are the Christoffel symbols. We will consider the above action in the latter choice or "metric gauge." This is the first step in completing the above diagram.

The natural gauge transformations for this action are the diffeomorphisms acting on the Christoffel symbols. The Christoffel symbol transforms as.

$$\Gamma'(x')_{a}{}^{c}{}_{b} = g^{c}{}_{f}\ \Gamma(x)_{e}{}^{f}{}_{g}\ (g^{-1})^{e}{}_{a}\ (g^{-1})^{g}{}_{c} - (g^{-1})^{e}{}_{a}\ \partial_{d}(g^{b}{}_{e})\ (g^{-1})^{e}{}_{a}, \tag{2.4}$$

where g is defined as

$$(g)^{e}{}_{a} \ = \frac{\partial x'^{e}}{\partial x^{a}}. \tag{2.5}$$

Now, for simplicity, I will define $\Gamma^{a}{}_{b} = \Gamma_{c}{}^{a}{}_{b}\ dx^{c}$. The initial action now becomes

$$\mathbf{S}_1 = \frac{k}{4\pi}\int_{\mathrm{M}}\Big(\Gamma^{a}{}_{b}\ \wedge\ \mathrm{d}\Gamma^{b}{}_{a} - \frac{2}{3}\Gamma^{a}{}_{b}\ \wedge\ \Gamma^{b}{}_{c}\ \wedge\ \Gamma^{c}{}_{a}\Big). \tag{2.6}$$

The above gauge transformation (2.4) now appears as a normal gauge transformation.

$$\Gamma'(x')^{c}{}_{b} = (g^{-1})^{b}{}_{e}\ \Gamma(x)^{e}{}_{f}\ g^{f}{}_{c} + (g^{-1})^{b}{}_{e}\ \mathrm{d}(g)^{e}{}_{a} \tag{2.7}$$

The above action transforms under this gauge transformation as

$$\mathbf{S}_1(\Gamma') = \mathbf{S}_1(\Gamma) - \frac{k}{4\pi}\int_{\partial\mathrm{M}}\mathrm{Tr}\left(\left(\mathrm{d}g g^{-1}\right)\wedge\Gamma\right) - \frac{k}{12\pi}\int_{\mathrm{M}}\mathrm{Tr}\left(g^{-1}\mathrm{d}g\right)^{3} \tag{2.8}$$

If the manifold has no boundary the second term is zero, and the last term is just the winding number. Thus up to the winding number, this action on a manifold without boundary is invariant under this guage transformation. However, with the boundary, if we take the variational derivative to find the classical equation of motion, a boundary term also appears.

$$\delta\mathbf{S}_1 = \frac{k}{2\pi}\int_{\mathrm{M}}\mathrm{Tr}\left(\delta\Gamma\left(\mathrm{d}\Gamma + \Gamma\wedge\Gamma\right)\right) - \frac{k}{4\pi}\int_{\partial\mathrm{M}}\mathrm{Tr}\left(\Gamma\wedge\delta\Gamma\right) \tag{2.9}$$

To be able to find the extremum for the classical equation of motion, the boundary terms must be zero. We would like it to be zero without fixing all of the fields on the boundary or setting them equal to zero. This problem is equivalent to picking the necessary boundary data to solve the equations of motion. A standard choice in this system is to pick a complex structure on the boundary, and fix one component of the field, and then add a surface term to the original action to cancel the above boundary term. This choice of field fixing allows us to uniquely map solutions of the classical equations of motion on one manifold to another across a shared boundary. This allows sewing (or gluing) manifolds with boundaries together to create a new manifold with a well defined action (see [7]). In our case we will choose to fix $\Gamma_{z}{}^{a}{}_{b}$, and add the following boundary term for the action.

$$\mathbf{S}_2 = \frac{k}{4\pi}\int_{\partial\mathrm{M}}\Gamma_{\bar{z}}{}^{a}{}_{b}\Gamma_{z}{}^{b}{}_{a}dzd\bar{z} \tag{2.10}$$

Taking the variation of the resulting action we get

$$\delta(\mathbf{S}_1+\mathbf{S}_2)=\frac{k}{2\pi}\int_{\mathrm{M}}\mathrm{Tr}\left(\delta\Gamma\left(\mathrm{d}\Gamma+\Gamma\wedge\Gamma\right)\right)-\frac{k}{2\pi}\int_{\partial\mathrm{M}}\Gamma_{\bar{z}}{}^{a}{}_{b}\delta\Gamma_{z}{}^{b}{}_{a}dzd\bar{z}. \tag{2.11}$$

$\delta\Gamma_{z}{}^{a}{}_{b}$ is equal to zero, and the boundary term is zero. With this "fixed" action, the extremum now gives well defined equations of motion.

Considering the gauge transformation on this new action with the additional boundary term, we get

$$(\mathbf{S}_1+\mathbf{S}_2)[\Gamma']=(\mathbf{S}_1+\mathbf{S}_2)[\Gamma]+k\mathbf{S}^{+}_{WZW}[g,\Gamma] \tag{2.12}$$

$$\mathbf{S}^{+}_{WZW}[g,\Gamma]=\frac{1}{4\pi}\int_{\partial\mathrm{M}}\mathrm{Tr}\left(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}}-2g^{-1}\partial_z g\ \Gamma_{\bar{z}}\right)+\frac{1}{12\pi}\int_{\mathrm{M}}\mathrm{Tr}\left(g^{-1}\mathrm{d}g\right)^3. \tag{2.13}$$

I would like to consider this action in a path integral setting. To begin with, we can use the standard Fadeev-Popov gauge fixing methods to get the measure,

$$[\mathrm{d}\Gamma']=[\mathrm{d}\Gamma][\mathrm{d}g]\ \delta\left(F[\Gamma]\right)\Delta_F[\Gamma]. \tag{2.14}$$

F is our gauge fixing function $F(g)=0$. In the Fadeev-Popov method, Δ_F must be gauge invariant. In order for Δ_F to be gauge invariant $[\mathrm{d}g]$ must be a left invariant measure (see Ryder [8] for more details). The path integral now becomes

$$\mathbf{Z}=\left(\int[\mathrm{d}\Gamma]\delta\left(F[\Gamma]\right)\Delta[\Gamma]\exp\left\{i\mathbf{S}[\Gamma]\right\}\right)\left(\int[\mathrm{d}g]\exp\left\{ik\mathbf{S}^{+}_{WZW}[g,\Gamma_z]\right\}\right) \tag{2.15}$$

The gauge dependent piece factors out, and because $\Gamma_{\bar{z}}$ is fixed, these two pieces are independent of each other. The first term is the "bulk" term and can be ignored. The second term is just a WZW model coupled to a fixed back-ground.

A convenient choice of gauge would be to let $h^{ab}\Gamma_{a}{}^{c}{}_{b}=0$. This is a called the harmonic gauge (see ref [9]). The result of this gauge fixing leads to finding the determinant of the Lichnerowicz operator.

$$(g^{ab}(\nabla_a\nabla_b\delta^c_d)+\mathrm{R}^c_d)\delta^2(z-z') \tag{2.16}$$

However this is not needed to continue the analyzes of the WZW model.

3 Chiral Diffeomorphism and Liouville Theory

Following Carlip's paper [1], I will consider the chiral diffeomorphism $z\rightarrow w(z,\bar{z})$ and $\bar{z}\rightarrow\bar{z}$ (note: $g=g^{-1}$ in Carlip's paper). Let the metric on the original boundary be give by

$$\mathrm{d}s^2=e^{2\xi}\mathrm{d}z\mathrm{d}\bar{z}. \tag{3.1}$$

We see that Γ_z, the fixed field on ∂M, is

$$\Gamma_z = \begin{pmatrix} 2\partial_z\xi & 0 \\ 0 & 0 \end{pmatrix}. \tag{3.2}$$

Under this chiral diffeomorphism, the group element (2.5) may be written in the following way.

$$g = \begin{pmatrix} \dfrac{\partial w(z,\bar{z})}{\partial z} & \dfrac{\partial w(z,\bar{z})}{\partial \bar{z}} \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} e^{\phi} & \mu \\ 0 & 1 \end{pmatrix} \tag{3.3}$$

Because the partial derivatives commute, we should also include constraint equation,

$$\frac{\partial}{\partial \bar{z}}\left(\frac{\partial w}{\partial z}\right) = \frac{\partial}{\partial z}\left(\frac{\partial w}{\partial \bar{z}}\right) \quad \Rightarrow \quad \frac{\partial}{\partial \bar{z}}\left(e^{\phi}\right) = \frac{\partial \mu}{\partial z}. \tag{3.4}$$

This constraint was not included into the action in ref.[1], and thus the resulting field theory included no cosmological constant term.

Substituting (3.3) into $\mathbf{S}^{+}_{WZW}[g, \Gamma_z]$.

$$g^{-1}\mathrm{d}g = \begin{pmatrix} \mathrm{d}\phi & -\mu\mathrm{d}\phi \\ 0 & 0 \end{pmatrix} \tag{3.5}$$

$$\text{first term} \implies \mathrm{Tr}(g^{-1}\partial_z g g^{-1}\partial_{\bar{z}} g) = \partial_z\phi\partial_{\bar{z}}\phi \tag{3.6}$$

$$\text{second term} \implies \mathrm{Tr}(g^{-1}\partial_z g\Gamma_z) = 4\partial_{\bar{z}}\phi\partial_z\xi \tag{3.7}$$

Integrating by parts, we get

$$4\partial_{\bar{z}}\phi\partial_z\xi = 4\phi\partial_{\bar{z}}\partial_z\xi = \sqrt{g}R\phi. \tag{3.8}$$

$$\text{last term} \implies \mathrm{Tr}(g^{-1}\mathrm{d}g) = \partial_i\phi\partial_j\phi\partial_k\phi\epsilon^{ijk} = 0 \tag{3.9}$$

So all together we get the action,

$$\mathbf{S}[\phi] = \int (\partial_z\phi\partial_{\bar{z}}\phi + \sqrt{g}R\phi)dzd\bar{z}. \tag{3.10}$$

The choice of which boundary field is fixed determines how the gauge fields couple to the curvature R. If we choose to fix $\Gamma_{\bar{z}}$ with the above choice for the chiral gauge field, the gauge field would not couple to the curvature R. Instead (3.10) would be a free field action.

In the above action (3.10), there is no dependence on μ. However, we will see that there is μ dependence in the measure when we consider the constraint. We know the measure because it was induced from the original Chern-Simons measure (2.14). As mentioned before, in the standard Fadeev-Popov method, the gauge group measure must be left invariant. It can easily be shown that $[dg] = [e^{-\phi}d\phi][d\mu]$ is left invariant (where $[e^{-\phi}d\phi] = \mathrm{Det[e^{-\phi}]}[\mathrm{d}\phi]$ is

defined as in Reference [10]). We then need to include the constraint into the measure in a left invariant manner. The following is a left invariant form of the delta function, which we will include in the the measure.

$$\delta\left(\frac{\partial\phi}{\partial\bar{z}} - e^{-\phi}\frac{\partial\mu}{\partial z}\right) \tag{3.11}$$

Writing the delta function as a Fourier transform, we find the full left invariant measure as

$$[dg]_{left} = [e^{-\phi}d\phi][d\mu][d\lambda]\ \exp\left\{\ -i\lambda\int_{\partial \mathrm{M}}(\partial_{\bar{z}}\phi - e^{-\phi}\partial_z\mu)\right\}. \tag{3.12}$$

Now if we integrate out μ, we will get the delta function

$$\mathrm{Det}[\mathrm{e}^{\phi}]\ \delta(\partial_{\mathrm{z}}(\lambda \mathrm{e}^{-\phi})). \tag{3.13}$$

$\mathrm{Det}[\mathrm{e}^{\phi}]$ cancels with the like term in the original measure. Solving for when the delta function is non-zero we see that

$$\lambda = f(\bar{z})e^{\phi}, \tag{3.14}$$

where $f(\bar{z})$ is an arbitrary function of $\bar{z}$. Substituting this into the partition function and integrating by parts gives

$$\mathbf{Z} = \int [d\phi]\mathrm{Det}[\partial_z][df(\bar{z})]\ \exp\left\{\ \int_{\partial \mathrm{M}}(\partial_{\bar{z}}f(\bar{z})e^{\phi})\right\}\ \exp\left\{\ \mathbf{S}[\phi]\right\}. \tag{3.15}$$

Now let $\partial_{\bar{z}}f(\bar{z}) = \mu_0 e^{\phi'(\bar{z})}$ and $\phi \to \phi + \phi'(\bar{z})$. Then terms in the action that depend on $\phi'(\bar{z})$ can be written as complete derivatives including the term coupling ϕ and R. This is easiest to see by looking back to (3.7). The volume term $Det[\partial_z][df(\bar{z})]$ can be integrated out. The final partition function is given by

$$\mathbf{Z} = \int [d\phi]e^{S_L[\phi,\mu_0]},$$

$$\text{where} \qquad \mathbf{S}_L = \int \partial_{\bar{z}}\phi\partial_z\phi + \mu_0 e^{\phi} + \sqrt{g}R\phi \tag{3.16}$$

which is the Liouville action. So we now have come full circle in the above diagram.

4 Discussion

It is clear that a chiral diffeomorphism is not the full diffeomorphism "gauge" group. The full group is difficult to deal with because it can not be simply factored like ordinary gauge groups. Thus, we can not use the standard Polyakov and Wiegman factorization [11] to simplify the math. We can see this in the following way. Although the chain rule lets us multiply two elements to get a new element, the second element below is not in the original coordinate system.

$$g(z,\bar{z}) = g_2(u(z,\bar{z}),\bar{u}(z,\bar{z}))\; g_1(z,\bar{z}) \tag{4.1}$$

$$\begin{tikzcd} & (u,\bar{u}) \arrow[dr, "g_2"] & \\ (z,\bar{z}) \arrow[ur, "g_1"] \arrow[rr, "g"] & & (w,\bar{w}) \end{tikzcd} \tag{4.2}$$

The above forms a semi-direct product because of the dependence of u on the g_1 transformation. Thus we can not factor it into a direct product and use a Gaussian decomposition of the group that is used in standard WZW models.

However, one can look at additional "conformal" diffeomorphisms. Consider the transformation

$$g'(z,\bar{z}) = a(\bar{v})g(u,\bar{u})b(z) \tag{4.3}$$

$$(z,\bar{z}) \xrightarrow{\;b\;} (u,\bar{u}) \xrightarrow{\;g\;} (v,\bar{v}) \xrightarrow{\;a\;} (w,\bar{w}).$$

$g(z,\bar{z})$ is our chiral diffeomorphism given in (3.3). The $b(z)$ and $a(\bar{v})$ terms simply re-scale the z and $\bar{z}$ coordinates (ie, conformal transformations) and therefore can be dealt with easily because they preserve the complex structure. For the $a(\bar{v})$, we write out the chain rule and get

$$g = \begin{pmatrix} \dfrac{\partial w}{\partial z} & \dfrac{\partial w}{\partial \bar{z}} \\ 0 & \dfrac{\partial \bar{w}}{\partial \bar{z}} \end{pmatrix} = \begin{pmatrix} e^{\phi} & \mu \\ 0 & e^{\psi} \end{pmatrix}. \tag{4.4}$$

The ψ term comes directly form $a(\bar{v})$. This term does not not couple with the curvature term. Therefore, it is a free field action. The constraint $\partial_z(e^{\psi}) = 0$ then removes it. For $b(z)$, we will get $\phi \rightarrow \phi + \phi'(z)$. Like before, $\phi'(z)$ term can be written has a total derivative and can be dropped from the action. So this action is invariant in a similar way to $\mathbf{S}[a(\bar{z})\; g\; b(z)] = \mathbf{S}[g]$ in [11] and [12].

Unlike the earlier methods, this method will lead to an action that will couple to the curvature as well as the cosmological constant. Perhaps a full treatment of the diffeomorphism group will give more information about Liouville theory.

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